Perturbative calculation of one-point functions of one-dimensional single-species reaction-diffusion systems

Mohammad Khorrami* and Amir Aghamohammadi†
Department of Physics, Alzahra University, Tehran 19938-91167, Iran
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Perturbations around autonomous one-dimensional single-species reaction-diffusion systems are investigated. It is shown that the parameter space corresponding to the autonomous systems is divided into two parts: In one part, the system is stable against the perturbations, in the sense that the largest relaxation time of the one-point functions changes continuously with perturbations. In the other part, however, the system is unstable against perturbations, so that any small perturbation drastically modifies the large-time behavior of the one-point functions.

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I. INTRODUCTION

Reaction-diffusion systems are a well-studied area. Reaction-diffusion systems, using analytical techniques, approximation methods, and simulation, have been studied. The approximation methods may be different in different dimensions, such as, for example, mean-field techniques, are good for high dimensions, and generally do not give correct results for low-dimensional systems. A large fraction of analytical studies belong to low-dimensional (especially one-dimensional) systems, as solving low-dimensional systems should in principle be easier [1–13].

One of the reasons scientists want to find and solve exactly solvable systems is that one can use perturbative methods to investigate other systems, which are not exactly solvable but *near* some exactly solvable systems. The term "exactly solvable" has been used with different meanings. For example, in [14–16], solvability (or integrability) means that the N-particle conditional probabilities' S matrix is factorized into a product of two-particle S matrices, while in [17–26], solvability means closedness of the evolution equation of the empty intervals (or their generalization).

In [27], a ten-parameter family of reaction-diffusion processes was introduced for systems among which the evolution equation of n-point functions contains only n- or fewer-point functions. We call such systems autonomous. There, for these models the average particle number in each site was obtained exactly. In [28,29], this has been generalized to multispecies systems and more-than-two-site interactions.

Among the important aspects of reaction-diffusion systems is the phase structure of the system. The static phase structure concerns the time-independent profiles of the system, while the dynamical phase structure concerns the evolution of the system, especially its relaxation behavior. In [30–33], the phase structures of some classes of single- or multiple-species autonomous reaction-diffusion systems have been investigated. These investigations were based on the one-point functions of the systems.

In [34], the two-point functions of autonomous single-species translationally invariant one-dimensional reaction-diffusion systems were studied. The two-point function for such systems was obtained, and it was shown that it exhibits a nontrivial dynamical phase structure. The dynamical phase structure of the system was also investigated.

In this article, we want to use perturbation to study systems which are not exactly autonomous, but *nearly* autonomous. By this, it is meant that the rates of these systems are equal to those of an autonomous system, plus a small perturbation. The scheme of the paper is the following. In Sec. II, autonomous systems are briefly introduced. In Sec. III, nonautonomous perturbations around autonomous systems are considered, and their effect on the evolution of one-point functions is investigated. From this investigation, it turns out that some of the autonomous systems are unstable with respect to perturbations, in the sense that any small perturbation drastically modifies the large-time relaxation of the one-point functions. Section IV is devoted to a concrete example.

II. AUTONOMOUS SYSTEMS AND THE EVOLUTION EQUATIONS

To fix the notation, let us briefly introduce the autonomous systems. Consider a one-dimensional periodic lattice, every point of which either is empty or contains one particle. Let the lattice have $L+1$ sites. The observables of such a system are the operators N_i^α, where i with $1 \leqslant i \leqslant L+1$ denotes the site number, and $\alpha = 0, 1$ denotes the hole or the particle: N_i^0 is the hole (vacancy) number operator at site i, and N_i^1 is the particle number operator at site i. One has obviously the constraint

$$s_\alpha N_i^\alpha = 1, \tag{1}$$

where **s** is a covector the components of which (s_α's) are all equal to 1. The constraint (1) simply says that every site either is occupied by one particle or is empty. A representation for these observables is

$$N_i^\alpha := \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes N^\alpha \otimes \underbrace{1 \otimes \cdots \otimes 1}_{L+1-i}, \tag{2}$$

*Electronic address: mamwad@mailaps.org
†Electronic address: mohamadi@azzahra.ac.ir

where N^α is a diagonal 2×2 matrix the only nonzero element of which is the αth diagonal element, and the operators 1 in the above expression are also 2×2 matrices. It is seen that the constraint (1) can be written as

$$\mathbf{s}\cdot\mathbf{N}=1, \tag{3}$$

where $\mathbf{N}$ is a vector the components of which are N^α's. The state of the system is characterized by a vector

$$\mathbb{P}\in\underbrace{\mathbb{V}\otimes\cdots\otimes\mathbb{V}}_{L+1}, \tag{4}$$

where $\mathbb{V}$ is a two-dimensional vector space. All the elements of the vector $\mathbf{P}$ are non-negative, and

$$\mathbf{S}\cdot\mathbf{P}=1. \tag{5}$$

Here $\mathbf{S}$ is the tensor product of $L+1$ covectors $\mathbf{s}$.

As the eigenvalues of the number operators N_i^α are 0 or 1 (and hence these operators are idempotent), the most general observable of such a system is the product of some of these number operators or a sum of such terms. Also, the constraint (1) shows that the two components of $\mathbf{N}_i$ are not independent. So one can express any function of $\mathbf{N}_i$ in terms of

$$n_i := \mathbf{a}\cdot\mathbf{N}_i, \tag{6}$$

where $\mathbf{a}$ is an arbitrary covector not parallel to $\mathbf{s}$. Our aim is to study the evolution of the one-point functions ($\langle n_i\rangle$'s).

The evolution of the state of the system is given by

$$\dot{\mathbf{P}}=\mathcal{H}\,\mathbf{P}, \tag{7}$$

where the Hamiltonian $\mathcal{H}$ is stochastic, by which it is meant that its nondiagonal elements are non-negative and

$$\mathbf{S}\,\mathcal{H}=0. \tag{8}$$

The interaction is nearest neighbor iff the Hamiltonian is of the form

$$\mathcal{H}=\sum_{i=1}^{L+1}H_{i,i+1}, \tag{9}$$

where

$$H_{i,i+1}:=\underbrace{1\otimes\cdots\otimes 1}_{i-1}\otimes H\otimes\underbrace{1\otimes\cdots\otimes 1}_{L-i}. \tag{10}$$

[It has been assumed that the sites of the system are identical—that is, that the system is translation invariant. Otherwise, H in the right-hand side of Eq. (10) would depend on i.] The two-site Hamiltonian H is stochastic; that is, its nondiagonal elements are non-negative, and the sum of the elements of each of its columns vanishes:

$$(\mathbf{s}\otimes\mathbf{s})H=0. \tag{11}$$

Using

$$\mathbf{s}\otimes\mathbf{s}(\mathbf{a}\cdot\mathbf{N})\otimes(\mathbf{b}\cdot\mathbf{N})H=a_\alpha b_\beta H^{\alpha\beta}{}_{\gamma\delta}\mathbf{s}\otimes\mathbf{s}N^\gamma\otimes N^\delta, \tag{12}$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary covectors, one can write down the evolution equations of the one-, two-, or morepoint functions of n_i's. The evolution equation for the one-point function is

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle n_i\rangle=a_\alpha\, s_\beta\, H^{\alpha\beta}{}_{\gamma\delta}\,\langle N_i^\gamma\, N_{i+1}^\delta\rangle+s_\alpha\, a_\beta\, H^{\alpha\beta}{}_{\gamma\delta}\,\langle N_{i-1}^\gamma\, N_i^\delta\rangle. \tag{13}$$

It is seen that the right-hand side of the above equation contains two-point functions. In fact, in the evolution equation of n-point functions, there are generally up to $(n+1)$-point functions. However, there are systems for them in the evolution equation of n-point functions, only up to n-point functions arise. These are the autonomous systems. For a system with the Hamiltonian H^0 to be autonomous, certain constraints should hold among the reaction rates [27–29]:

$${}^e\mathcal{A}^{0\alpha}{}_{\gamma\delta}={}^e_1\mathcal{A}^{0\alpha}{}_\gamma\, s_\delta+{}^e_2\mathcal{A}^{0\alpha}{}_\delta s_\gamma, \tag{14}$$

where

$${}^1\mathcal{A}^{0\alpha}{}_{\gamma\delta}:=s_\beta\, H^{0\alpha\beta}{}_{\gamma\delta},$$

$${}^2\mathcal{A}^{0\alpha}{}_{\gamma\delta}:=s_\beta\, H^{0\beta\alpha}{}_{\gamma\delta}. \tag{15}$$

The constraints mean that the left-hand side of Eq. (14) can be decomposed as the right-hand side of Eq. (14). It is not difficult to see that the constraints (14) are equivalent to

$$H^0\,\mathbf{u}\otimes\mathbf{u}=\lambda\,\mathbf{u}\otimes\mathbf{u}, \tag{16}$$

where

$$\mathbf{u}:=\begin{pmatrix}1\\-1\end{pmatrix}, \tag{17}$$

and it is obvious that

$$\mathbf{s}\cdot\mathbf{u}=0. \tag{18}$$

Now, consider an autonomous system satisfying the constraints (14) [or equivalently (16)], and take the vector $\mathbf{v}$ satisfying

$$\left(\sum_{d,e=1}^{2}{}^d_e\mathcal{A}^0\right)\mathbf{v}=0,$$

$$\mathbf{s}\cdot\mathbf{v}=1, \tag{19}$$

and the covector $\mathbf{a}$ such that

$$\mathbf{a}\cdot\mathbf{u}=1,\quad \mathbf{a}\cdot\mathbf{v}=0; \tag{20}$$

that is, the basis $\{\mathbf{a},\mathbf{s}\}$ is dual to $\{\mathbf{u},\mathbf{v}\}$. In [28,29], it is shown that the matrix on the left-hand side of the first equation in Eqs. (19) has a left eigenvector with the eigenvalue zero. (This left eigenvector is $\mathbf{s}$.) So it does have a right eigenvector with the eigenvalue zero as well. That is, there does exist a vector $\mathbf{v}$ satisfying Eqs. (19). In fact, one can even find a real vector $\mathbf{v}$ satisfying Eqs. (19). From now on, $\mathbf{a}$ in Eq. (6) is assumed to satisfy Eqs. (20).

III. PERTURBATIONS AROUND AUTONOMOUS SYSTEMS

Consider a system with the Hamiltonian H as

$$H = H^0 + \delta H, \tag{21}$$

where H^0 is the Hamiltonian of an autonomous system, and the rates corresponding to δH are small compared to those corresponding to H^0. Our task is to investigate the evolution of one-point functions corresponding to H, using perturbation. As H is not necessarily autonomous, the evolution equation of the one-point function may contain two-point functions as well. However, the terms containing the two-point functions are proportional to the rates corresponding to δH and hence are small. So one can calculate the two-point function corresponding to the unperturbed system and use it in the evolution equation of the one-point function of the perturbed system to obtain up-to-first-order evolution of the one-point function of the perturbed system.

A. Unperturbed solution

Assuming that the initial condition is translationally invariant, it is seen that the one-point function is independent of the site and the two-point function depends on only the difference of the sites' numbers. So the evolution equation for the one-point function of the unperturbed system is

$$\frac{df^0}{dt} = (\mu^0 + \nu^0) f^0, \tag{22}$$

where

$$f^0 := \langle n_i \rangle^0 \tag{23}$$

and

$$\mu^0 = \mathbf{s} \otimes \mathbf{a}\ H^0\ \mathbf{u} \otimes \mathbf{v} + \mathbf{a} \otimes \mathbf{s}\ H^0\ \mathbf{v} \otimes \mathbf{u},$$

$$\nu^0 = \mathbf{s} \otimes \mathbf{a}\ H^0\ \mathbf{v} \otimes \mathbf{u} + \mathbf{a} \otimes \mathbf{s}\ H^0\ \mathbf{u} \otimes \mathbf{v}. \tag{24}$$

The one-point function f^0 is easily seen to be

$$f^0(t) = f^0(0)\exp[(\mu^0 + \nu^0)t]. \tag{25}$$

Also, taking

$$F_i^0 := \langle n_k\ n_{k+i} \rangle^0 \tag{26}$$

(the two-point function of the unperturbed system), one arrives at

$$\frac{dF_i^0}{dt} = \mu^0 (F_{i-1}^0 + F_{i+1}^0) + 2\nu^0\ F_i^0, \quad 1 < i < L,$$

$$\frac{dF_1^0}{dt} = \mu^0\ F_2^0 + (\nu^0 + \lambda^0) F_1^0 + \rho^0\ f^0 + \sigma^0, \tag{27}$$

where

$$\lambda^0 := \mathbf{a} \otimes \mathbf{a}\ H^0\ \mathbf{u} \otimes \mathbf{u},$$

$$\rho^0 := \mathbf{a} \otimes \mathbf{a}\ H^0 (\mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u}),$$

$$\sigma^0 := \mathbf{a} \otimes \mathbf{a}\ H^0\ \mathbf{v} \otimes \mathbf{v}. \tag{28}$$

It is seen that only five parameters enter the evolution equation of the up-to-two-point functions, and all of these can be expressed in terms of the matrix elements of

$$\overline{H^0} := H^0 + \Pi\ H^0\ \Pi, \tag{29}$$

where Π is the permutation matrix. These parameters can be rewritten as

$$\mu^0 := \mathbf{s} \otimes \mathbf{a}\ \overline{H^0}\ \mathbf{u} \otimes \mathbf{v},$$

$$\nu^0 := \mathbf{s} \otimes \mathbf{a}\ \overline{H^0}\ \mathbf{v} \otimes \mathbf{u},$$

$$\lambda^0 := \frac{1}{2}\mathbf{a} \otimes \mathbf{a}\ \overline{H^0}\ \mathbf{u} \otimes \mathbf{u},$$

$$\rho^0 := \mathbf{a} \otimes \mathbf{a}\ \overline{H^0}\ \mathbf{u} \otimes \mathbf{v},$$

$$\sigma^0 := \frac{1}{2}\mathbf{a} \otimes \mathbf{a}\ \overline{H^0}\ \mathbf{v} \otimes \mathbf{v}. \tag{30}$$

Taking a solution like

$$F_i^0(t) = \sum_{E^0} F_{i\ E^0}^0(0)\exp(E^0 t), \tag{31}$$

it was shown in [34] that the values of E^0 (energy values) entering the two-point function are 0, $E_1^0 := \mu^0 + \nu^0$, and any number in the interval $I_0 := [2\nu^0 - 2|\mu^0|, 2\nu^0 + 2|\mu^0|]$, and possibly

$$E_2^0 := \lambda^0 + \nu^0 + \frac{(\mu^0)^2}{\lambda^0 - \nu^0}. \tag{32}$$

E_2^0 is among the possible values of E^0 iff

$$|\mu^0| \leqslant \lambda^0 - \nu^0. \tag{33}$$

The relation of E_1^0, E_2^0, and I_0 determines the relaxation behavior of the two-point function (its dynamical phase). Depending on the reaction rates several phases may occur [34]:

(I) $E_1^0 \in I_0$, and E_2^0 is not an energy. This is the slower phase, and the longest relaxation time is $[-2(\nu^0 - \mu^0)]^{-1}$.

(II) $E_1^0 \in I_0$, and E_2^0 is an energy, in fact the largest one. This is the slowest phase, and the longest relaxation time is $\{-\nu^0 - \lambda^0 - [(\mu^0)^2/(\lambda^0 - \nu^0)]\}^{-1}$.

(III) $E_1^0 > I_0$, and E_2^0 is not an energy. This is the fastest phase, and the longest relaxation time is $[-(\nu^0 + \mu^0)]^{-1}$.

(IV) $E_1^0 > I_0$, E_2^0 is an energy, and $E_2^0 < E_1^0$. This is the fast phase, and the longest relaxation time is $[-(\nu^0 + \mu^0)]^{-1}$.

(V) $E_1^0 > I_0$, E_2^0 is an energy, and $E_2^0 > E_1^0$. This is the slow phase, and the longest relaxation time is $\{-\nu^0 - \lambda^0 - [(\mu^0)^2/(\lambda^0 - \nu^0)]\}^{-1}$.

B. Perturbed solution

Now consider the Hamiltonian H, defined through Eq. (21), which is not necessarily autonomous. Then defining ${}^e\mathcal{A}_{\gamma\delta}$ like Eqs. (15) but with H instead of H^0, it is seen that the evolution equation of the one-point function is

$$\frac{\mathrm{d}\langle n_i\rangle}{\mathrm{d}t} = a_\alpha\, {}^1\mathcal{A}^\alpha{}_{\gamma\delta}\langle N_i^\gamma N_{i+1}^\delta\rangle + a_\alpha\, {}^2\mathcal{A}^\alpha{}_{\gamma\delta}\langle N_{i-1}^\gamma N_i^\delta\rangle. \quad (34)$$

However, one cannot necessarily decompose ${}^e\mathcal{A}^\alpha{}_{\gamma\delta}$ like Eq. (14). Assuming translational invariance of the initial conditions, one arrives at

$$\frac{\mathrm{d}f}{\mathrm{d}t} = a_\alpha({}^1\mathcal{A}^\alpha{}_{\gamma\delta} + {}^2\mathcal{A}^\alpha{}_{\gamma\delta})F_1^{\gamma\delta}, \quad (35)$$

where

$$f := \langle n_i\rangle,$$

$$F_1^{\gamma\delta} := \langle N_i^\gamma N_{i+1}^\delta\rangle. \quad (36)$$

As $\{\mathbf{u},\mathbf{v}\}$ is a basis, one can write F_1 in terms of the tensor products of $\mathbf{u}$ and $\mathbf{v}$. The corresponding coefficients can be found by multiplying the tensor products of $\mathbf{a}$ and $\mathbf{s}$ by F_1. The result is

$$F_1^{\gamma\delta} = F_1 u^\gamma u^\delta + f(u^\gamma v^\delta + v^\gamma u^\delta) + v^\gamma v^\delta. \quad (37)$$

So

$$\frac{\mathrm{d}f}{\mathrm{d}t} = a_\alpha({}^1\mathcal{A}^\alpha{}_{\gamma\delta} + {}^2\mathcal{A}^\alpha{}_{\gamma\delta})[F_1 u^\gamma u^\delta + f(u^\gamma v^\delta + v^\gamma u^\delta) + v^\gamma v^\delta]. \quad (38)$$

Defining

$${}^e\mathcal{B}^\alpha{}_{\gamma\delta} := {}^e\mathcal{A}^\alpha{}_{\gamma\delta} - {}^e\mathcal{A}^{0\alpha}{}_{\gamma\delta}, \quad (39)$$

Eq. (38) recasts to

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= a_\alpha\big({}^1_1\mathcal{A}^{0\alpha}{}_\gamma\, s_\delta + {}^1_2\mathcal{A}^{0\alpha}{}_\delta\, s_\gamma + {}^2_1\mathcal{A}^{0\alpha}{}_\gamma\, s_\delta + {}^2_2\mathcal{A}^{0\alpha}{}_\delta\, s_\gamma + {}^1\mathcal{B}^\alpha{}_{\gamma\delta} \\ &\quad + {}^2\mathcal{B}^\alpha{}_{\gamma\delta}\big)[F_1 u^\gamma u^\delta + f(u^\gamma v^\delta + v^\gamma u^\delta) + v^\gamma v^\delta] \\ &= (\mu^0+\nu^0)f + ({}^1B_{\gamma\delta} + {}^2B_{\gamma\delta})[F_1 u^\gamma u^\delta + f(u^\gamma v^\delta + v^\gamma u^\delta) \\ &\quad + v^\gamma v^\delta], \end{aligned} \quad (40)$$

where

$${}^e\mathcal{B}_{\gamma\delta} := a_\alpha\, {}^e\mathcal{B}^\alpha{}_{\gamma\delta}. \quad (41)$$

As expected, the coefficients of F_1 on the right-hand side are small (first order in terms of the perturbation δH). So one can use the zeroth-order value of F_1 on the right-hand side to obtain the first-order value of f.

From Eqs. (25) and (27), it is seen that if $E_1^0 > I_0$, then one can write

$$F_1^0 = \hat{F}_1^0 + \frac{\rho^0}{\mu^0 - \lambda^0 - \mu^0 z} f^0, \quad (42)$$

where z satisfies

$$\mu^0 + \nu^0 = \mu^0(z + z^{-1}) + 2\nu^0, \quad (43)$$

and its modulus is less than 1, and $\hat{F}_1^0$ is like Eq. (31), but without a term corresponding to the energy value $E_1^0 = \mu^0 + \nu^0$. So one can write Eq. (40) as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = (\mu^0+\nu^0)f + ({}^1B_{\gamma\delta} + {}^2B_{\gamma\delta})\left(\frac{\rho^0}{\mu^0 - \lambda^0 - \mu^0 z} u^\gamma u^\delta + u^\gamma v^\delta + v^\gamma u^\delta\right) f + ({}^1B_{\gamma\delta} + {}^2B_{\gamma\delta})(\hat{F}_1 u^\gamma u^\delta + v^\gamma v^\delta). \quad (44)$$

This means that the energy values entering f are those entering $\hat{F}_1^0$ and

$$E_1 := \mu + \nu + \frac{\rho^0}{\mu^0 - \lambda^0 - \mu^0 z}\delta\theta, \quad (45)$$

where

$$\begin{aligned}\mu &:= \mu^0 + \delta\mu, \\ \nu &:= \nu^0 + \delta\nu, \\ \delta\mu &:= \mathbf{s}\otimes\mathbf{a}\ \overline{\delta H}\mathbf{u}\otimes\mathbf{v}, \\ \delta\nu &:= \mathbf{s}\otimes\mathbf{a}\ \overline{\delta H}\mathbf{v}\otimes\mathbf{u}, \\ \delta\theta &:= \mathbf{a}\otimes\mathbf{s}\ \overline{\delta H}\mathbf{u}\otimes\mathbf{u}, \end{aligned} \quad (46)$$

and

$$\overline{\delta H} := \delta H + \Pi\ \delta H\ \Pi. \quad (47)$$

This shows that if E_1^0 is the largest nonzero energy value entering F_1^0, then E_1 is the largest nonzero energy value entering f. Otherwise, the largest nonzero energy value entering f is the largest nonzero energy value entering F_1^0. So the relaxation behavior of f can be deduced from that of F_1^0 as follows.

(I) In this phase the largest nonzero energy value of f is $2(\nu^0 - \mu^0)$, and the perturbation causes a discontinuous change of the largest nonzero energy value, from $E_1^0 = \mu^0 + \nu^0$ to $2(\nu^0 - \mu^0)$.

(II, V) In this phase the largest nonzero energy value of f is E_2^0, and the perturbation causes a discontinuous change of the largest nonzero energy value, from $E_1^0 = \mu^0 + \nu^0$ to E_2^0.

(III, IV) In this phase the largest nonzero energy value of f is E_1, and the perturbation causes a continuous change of the largest nonzero energy value, from E_1^0 to E_1.

It is seen that the perturbation causes two different changes in the relaxation behavior of f (the one-point function). In regions I, II, and V, the perturbation causes a discontinuous change in the relaxation behavior, which means that the autonomous system is unstable with respect to perturbations. In regions III and IV, however, the relaxation behavior of f is continuous with respect to the perturbations, which means that the autonomous system is stable with respect to the perturbations, at least as long as first-order perturbations of the one-point function are considered. Mentioning one other thing is also in order: F_1^0 enters the evolution equation of f iff $\delta\theta \neq 0$. If $\delta\theta = 0$, then f would contain only one (nonzero) energy value, which is the one expected from changing one autonomous system to another. (Only μ^0 is replaced by μ and ν^0 is replaced by ν.)

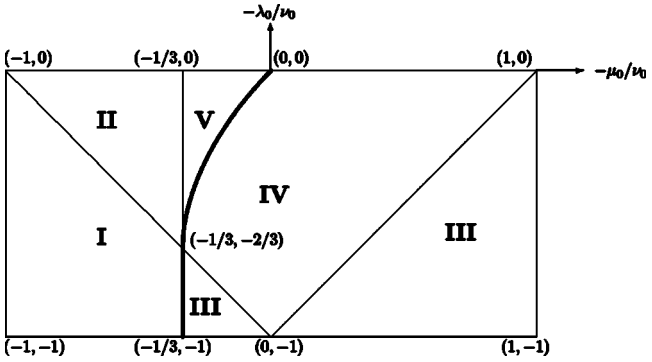

FIG. 1. Dynamical phase structure in the plane $(-\mu^0/\nu^0, -\lambda^0/\nu^0)$. The thick line is the boundary between effectively autonomous systems (right side of the boundary) and effectively nonautonomous systems (left side of the boundary).

A real autonomous system would in fact be only approximately autonomous. This means that there are always perturbations around the autonomous system. The above argument shows that only autonomous systems in regions III and IV can be effectively autonomous. The parameter space corresponding to the effectively autonomous systems is

$$-\frac{\mu^0}{\nu^0} > -\frac{1}{3},$$
$$-\frac{\lambda^0}{\nu^0} > \frac{(-\mu^0/\nu^0) - 1 + \sqrt{[1+3(-\mu^0/\nu^0)][1-(-\mu^0/\nu^0)]}}{2}. \tag{48}$$

Figure 1 shows the regions corresponding to the autonomous system. (This is identical to Fig. 1 in [34].)

IV. EXAMPLE

Consider a Hamiltonian H^0 corresponding to an autonomous system:

$$H^0 = \frac{1}{4}\begin{pmatrix} -3+3\omega & \omega & \omega & 1-\omega \\ 1-\omega & -3\omega & \omega & 1-\omega \\ 1-\omega & \omega & -3\omega & 1-\omega \\ 1-\omega & \omega & \omega & -3+3\omega \end{pmatrix} + r\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2 \end{pmatrix}. \tag{49}$$

The reactions of the corresponding system are

$$\varnothing A \rightarrow \text{any other state}, \quad \text{with the rate } \omega/4,$$
$$A\varnothing \rightarrow \text{any other state}, \quad \text{with the rate } \omega/4,$$
$$\varnothing\varnothing \rightarrow \varnothing A, \quad \text{with the rate } r+[(1-\omega)/4],$$
$$\varnothing\varnothing \rightarrow A\varnothing, \quad \text{with the rate } r+[(1-\omega)/4],$$
$$\varnothing\varnothing \rightarrow AA, \quad \text{with the rate} (1-\omega)/4,$$
$$AA \rightarrow \varnothing A, \quad \text{with the rate} (1-\omega)/4,$$
$$AA \rightarrow A\varnothing, \quad \text{with the rate} (1-\omega)/4,$$
$$AA \rightarrow \varnothing\varnothing, \quad \text{with the rate } r+[(1-\omega)/4]. \tag{50}$$

For this Hamiltonian, one has

$$\mathbf{v} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$\mathbf{a} = \frac{1}{2}(1 \quad 1), \tag{51}$$

and

$$\mu^0 = -1-2r+2\omega,$$
$$\nu^0 = -1-2r,$$
$$\lambda^0 = -\frac{1}{2} - r,$$
$$\rho^0 = r. \tag{52}$$

For the perturbation, consider the Hamiltonian

$$\delta H = \varepsilon\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{53}$$

It is easy to see that this Hamiltonian does not correspond to an autonomous system. This Hamiltonian only increases the rate of the reaction $AA \rightarrow \varnothing\varnothing$ by ε. Using this Hamiltonian, it is seen that

$$\delta\mu = -\varepsilon,$$
$$\delta\nu = -\varepsilon,$$
$$\delta\theta = -2\varepsilon. \tag{54}$$

From Eqs. (52), it is seen that

$$-\frac{\mu^0}{\nu^0} = -1 + 2\frac{\omega}{1+2r},$$
$$-\frac{\lambda^0}{\nu^0} = -\frac{1}{2}. \tag{55}$$

Comparing this with Eq. (48), it is seen that the system is effectively autonomous iff

$$\omega > \frac{5-\sqrt{5}}{8}(1+2r). \tag{56}$$

As a special case of the above example, let us put $\omega = 1$. In this case, for the nonperturbed system we have the reactions

$$\begin{aligned}
\varnothing A &\rightarrow \text{any other state}, \quad \text{with the rate } 1/4,\\
A\varnothing &\rightarrow \text{any other state}, \quad \text{with the rate } 1/4,\\
\varnothing\varnothing &\rightarrow \varnothing A, \quad \text{with the rate } r,\\
\varnothing\varnothing &\rightarrow A\varnothing, \quad \text{with the rate } r,\\
AA &\rightarrow \varnothing\varnothing, \quad \text{with the rate } r.
\end{aligned} \tag{57}$$

In this case, the system is effectively autonomous iff

$$r < r_0 := \frac{1}{2} + \frac{1}{\sqrt{5}}. \tag{58}$$

It is seen that changing the value of r from 0 to $+\infty$, the system starts from phase III, passes through phases IV, V, and II, and finally reaches phase I. At $r=r_0$, the system goes from phase IV to phase V, which means that the system is no longer effectively autonomous.

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